

simple, namely, that factorisation is in general facilitated when one of the factors is known *à priori*.

Our present purpose is to exemplify and enforce this, the determinants taken for factorisation being, not 4-line minors of the 3rd compound as just spoken of, but for the sake of brevity 3-line minors of the 4th compound.*

(3) By the elimination of u, v, w , the derived set of equations in x, y, z is

$$\left. \begin{aligned} &|a_1b_2c_3d_4|x + |a_1b_2c_3d_5|y + |a_1b_2c_3d_6|z = 0 \\ &|a_1b_2c_3e_4|x + |a_1b_2c_3e_5|y + |a_1b_2c_3e_6|z = 0 \\ &\dots\dots\dots \\ &|c_1d_2e_3f_4|x + |c_1d_2e_3f_5|y + |c_1d_2e_3f_6|z = 0 \end{aligned} \right\}.$$

These are in number 15, *i. e.* $C_{6,3}$, and the number of 3-line determinants got therefrom by the elimination of x, y, z is thus 455, *i. e.* $C_{15,3}$.

(4) Probably the best and most expeditious way of dealing with them is by means of the Law of Complementaries. Writing for each element of the 15-by-3 array its complementary minor in $|a_1b_2c_3d_4e_5f_6|$ we obtain the companion array

$$\begin{array}{ccc} |e_5f_6| & |e_4f_6| & |e_4f_5| \\ |d_5f_6| & |d_4f_6| & |d_4f_5| \\ |d_5e_6| & |d_4e_6| & |d_4e_5| \\ |c_5f_6| & |c_4f_6| & |c_4f_5| \\ |c_5e_6| & |c_4e_6| & |c_4e_5| \\ |c_5d_6| & |c_4d_6| & |c_4d_5| \\ |b_5f_6| & |b_4f_6| & |b_4f_5| \\ |b_5e_6| & |b_4e_6| & |b_4e_5| \\ |b_5d_6| & |b_4d_6| & |b_4d_5| \\ |b_5c_6| & |b_4c_6| & |b_4c_5| \\ |a_5f_6| & |a_4f_6| & |a_4f_5| \\ |a_5e_6| & |a_4e_6| & |a_4e_5| \\ |a_5d_6| & |a_4d_6| & |a_4d_5| \\ |a_5c_6| & |a_4c_6| & |a_4c_5| \\ |a_5b_6| & |a_4b_6| & |a_4b_5|, \end{array}$$

and what we have got to do is the comparatively simple task of factorising its corresponding 455 3-line determinants, knowing that when we have done so we can pass with ease to the factorisation that is wanted.

(5) A little examination shows that the auxiliary determinants in question may be conveniently classified according to the number of letters involved in them, the smallest possible number being three and the greatest six.

* Probably the m th compound of $|a_{1n}|$ is most suitably denoted by $||a_{1n}||_m$; certainly nothing more helpful has as yet been proposed. In the same way the compounds just referred to may be denoted by $||a_1b_2c_3d_4e_5f_6||_3$ and $||a_1b_2c_3d_4e_5f_6||_4$, or even by $|\Delta_6|_3$ and $|\Delta_6|_4$.

In the case of *three* letters the determinant is simply an adjugate. We have, for example,

$$\begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | d_5 f_6 | & | d_4 f_6 | & | d_4 f_5 | \\ | d_5 e_6 | & | d_4 e_6 | & | d_4 e_5 | \end{vmatrix} = | d_4 e_5 f_6 |^2,$$

whence by taking complementaries we deduce the desired factorisation—

$$\begin{vmatrix} | a_1 b_2 c_3 d_4 | & | a_1 b_2 c_3 d_5 | & | a_1 b_2 c_3 d_6 | \\ | a_1 b_2 c_3 e_4 | & | a_1 b_2 c_3 e_5 | & | a_1 b_2 c_3 e_6 | \\ | a_1 b_2 c_3 f_4 | & | a_1 b_2 c_3 f_5 | & | a_1 b_2 c_3 f_6 | \end{vmatrix} = | a_1 b_2 c_3 |^2 \cdot \Delta_6,$$

where Δ_6 is written for $| a_1 b_2 c_3 d_4 e_5 f_6 |$.

The number of instances of this type of resolvability is manifestly $C_{6,3}$, i.e. 20.

(6) In the case of *four* letters it is necessary to make a subdivision. In the first place we have to consider the type

$$\begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | d_5 f_6 | & | d_4 f_6 | & | d_4 f_5 | \\ | c_5 f_6 | & | c_4 f_6 | & | c_4 f_5 | \end{vmatrix}$$

in which the four letters are so distributed that a selected one of the four appears in every row. Here the determinant vanishes, and the Law of Complementaries consequently gives us

$$\begin{vmatrix} | a_1 b_2 c_3 d_4 | & | a_1 b_2 c_3 d_5 | & | a_1 b_2 c_3 d_6 | \\ | a_1 b_2 c_3 e_4 | & | a_1 b_2 c_3 e_5 | & | a_1 b_2 c_3 e_6 | \\ | a_1 b_2 d_3 e_4 | & | a_1 b_2 d_3 e_5 | & | a_1 b_2 d_3 e_6 | \end{vmatrix} = 0,$$

the number of instances of the type being

$$C_{6,4} \times 4, \text{ i.e. } 60.$$

In the next place we have to consider auxiliaries such as

$$\begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | d_5 f_6 | & | d_4 f_6 | & | d_4 f_5 | \\ | c_5 e_6 | & | c_4 e_6 | & | c_4 e_5 | \end{vmatrix}, \quad \begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | c_5 f_6 | & | c_4 f_6 | & | c_4 f_5 | \\ | d_5 e_6 | & | d_4 e_6 | & | d_4 e_5 | \end{vmatrix},$$

in which each of two letters, *e* and *f* say, occurs twice, and the other two, *c* and *d* say, each once. The factorisation for both of these is the same—a fact worthy of separate note—namely,

$$| d_4 e_5 f_6 | \cdot | c_4 e_5 f_6 |;$$

and consequently we have

$$\begin{vmatrix} | a_1 b_2 c_3 d_4 | & | a_1 b_2 c_3 d_5 | & | a_1 b_2 c_3 d_6 | \\ | a_1 b_2 c_3 e_4 | & | a_1 b_2 c_3 e_5 | & | a_1 b_2 c_3 e_6 | \\ | a_1 b_2 d_3 f_4 | & | a_1 b_2 d_3 f_5 | & | a_1 b_2 d_3 f_6 | \end{vmatrix} = \dots = | a_1 b_2 c_3 | \cdot | a_1 b_2 d_3 | \cdot \Delta_6,$$

the number of instances of the type being

$$C_{6,2} \times C_{4,2} \times 2, \text{ i.e. } 180.$$

(7) If *five* letters are to be used, only one of them, *f* say, must occur twice; its companions must be two, *d* and *e* say, of the remaining five; and the third row must have two, *b* and *c* say, of the remaining three. The factorisation of such an auxiliary is

$$\begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | d_5 f_6 | & | d_4 f_6 | & | d_4 f_5 | \\ | b_5 c_6 | & | b_4 c_6 | & | b_4 c_5 | \end{vmatrix} = | d_4 e_5 f_6 | \cdot | b_4 c_5 f_6 |,$$

from which we obtain

$$\begin{vmatrix} | a_1 b_2 c_3 d_4 | & | a_1 b_2 c_3 d_5 | & | a_1 b_2 c_3 d_6 | \\ | a_1 b_2 c_3 e_4 | & | a_1 b_2 c_3 e_5 | & | a_1 b_2 c_3 e_6 | \\ | a_1 d_2 e_3 f_4 | & | a_1 d_2 e_3 f_5 | & | a_1 d_2 e_3 f_6 | \end{vmatrix} = | a_1 b_2 c_3 | \cdot | a_1 d_2 e_3 | \cdot \Delta_6,$$

the full number of the type being

$$C_{6,1} \times C_{5,2} \times C_{3,2}, \text{ i. e. } 180.$$

(8) When *all* the letters are used the type of auxiliary equality is

$$\begin{vmatrix} | e_5 f_6 | & | e_4 f_6 | & | e_4 f_5 | \\ | c_5 d_6 | & | c_4 d_6 | & | c_4 d_5 | \\ | a_5 b_6 | & | a_4 b_6 | & | a_4 b_5 | \end{vmatrix} = \begin{vmatrix} | c_4 e_5 f_6 | & | d_4 e_5 f_6 | \\ | a_4 b_5 c_6 | & | a_4 b_5 d_6 | \end{vmatrix},$$

whence we have

$$\begin{vmatrix} | a_1 b_2 c_3 d_4 | & | a_1 b_2 c_3 d_5 | & | a_1 b_2 c_3 d_6 | \\ | a_1 b_2 e_3 f_4 | & | a_1 b_2 e_3 f_5 | & | a_1 b_2 e_3 f_6 | \\ | c_1 d_2 e_3 f_4 | & | c_1 d_2 e_3 f_5 | & | c_1 d_2 e_3 f_6 | \end{vmatrix} = \begin{vmatrix} | a_1 b_2 d_3 | & | a_1 b_2 e_3 | \\ | d_1 e_2 f_3 | & | c_1 e_2 f_3 | \end{vmatrix} \cdot \Delta_6,$$

the resolvability in this case being only partial. The number of instances is readily seen to be 15.

(9) Summarising the results of this census of the resolvability of 455 3-line minors of the 4th compound of $| a_1 b_2 c_3 d_4 e_5 f_6 |$ we have—

$$\begin{array}{lll} 20 & \text{of the type} & | a_1 b_2 c_3 |^2 \cdot \Delta_6, \\ 60 & \text{,,} & 0, \\ 180 & \text{,,} & | a_1 b_2 c_3 | \cdot | a_1 b_2 d_3 | \cdot \Delta_6, \\ 180 & \text{,,} & | a_1 b_2 c_3 | \cdot | a_1 d_2 e_3 | \cdot \Delta_6, \\ 15 & \text{,,} & \begin{vmatrix} | a_1 b_2 d_3 | & | a_1 b_2 e_3 | \\ | d_1 e_2 f_3 | & | c_1 e_2 f_3 | \end{vmatrix} \cdot \Delta_6. \end{array}$$

In regard to the second of the six groups it must be carefully noted that the 0 (zero) given as the result is not $0 \cdot \Delta_6$, the truth being that the process of elimination by instalments here fails, and of course properly so, because the 0 arises from the fact of it being possible to choose from our set of 15 equations in x, y, z three equations that are not mutually independent—a possibility that in our original statement in § 2 was only hinted at. Further, it is worth noting that these determinants of zero value bear an external mark that fully warns us against accepting them as multiples of

the true eliminant like the others, namely, the entire absence of one of the six necessary letters.

(10) In the 4th compound of Δ_6 there are nineteen other arrays which are quite similar in extent and character to that dealt with in the foregoing, and which thus have the factorisation of their 3-line determinants equally fully known.

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